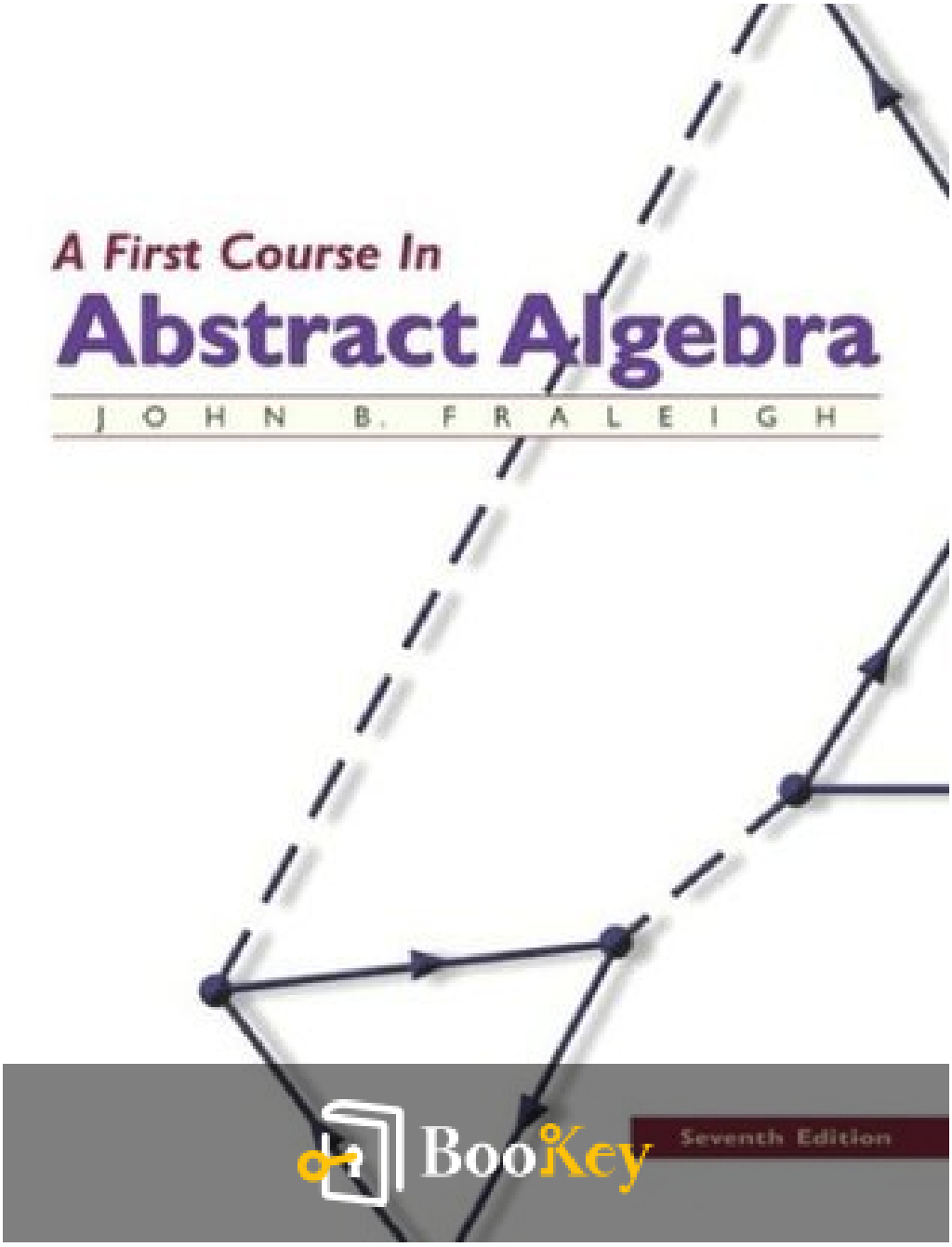
A First Course In Abstract Algebra PDF (Limited Copy)

John B. Fraleigh



A First Course In Abstract Algebra Summary

Building a Strong Foundation in Algebraic Structures and Concepts.

Written by New York Central Park Page Turners Books Club

About the book

Summary of "A First Course in Abstract Algebra"

This textbook serves as an essential introduction to the foundational concepts of abstract algebra, elegantly structured to promote a clear understanding of algebraic structures such as groups, rings, and fields. Each chapter is designed to build upon the previous material, establishing a logical progression that enriches students' comprehension of abstract mathematical theory.

Chapter 1: Introduction to Algebraic Structures

The book begins by introducing the notion of algebraic structures, delineating the importance of studying sets equipped with various operations. The chapter lays out the fundamental definitions and terminologies, including elements, operations, and axioms. It sets the stage for exploring how these structures interact and form the backbone of higher-level algebraic theory.

Chapter 2: Groups

Diving into groups, the text defines a group as a set accompanied by an operation that satisfies four key properties: closure, associativity, identity,

and invertibility. The chapter illustrates various examples of groups, including symmetric groups and cyclic groups, and discusses important concepts such as subgroup, cosets, and group homomorphisms. The significance of the Lagrange's theorem, which relates the size of a group to the size of its subgroups, is emphasized, solidifying students' understanding of the connection between group order and structure.

Chapter 3: Rings

Transitioning from groups to rings, the book introduces rings as algebraic structures that encompass two operations: addition and multiplication. It outlines the properties required for these operations, such as commutativity, distributivity, and the existence of additive inverses. Various types of rings, including commutative rings and integral domains, are discussed alongside key concepts like ideals and quotient rings. The chapter highlights the essential role of rings in connecting algebra to number theory and geometry.

Chapter 4: Fields

Next, the text focuses on fields, which can be viewed as rings with additional properties that allow for division. A field requires every non-zero element to have a multiplicative inverse, promoting a deeper exploration of algebraic structures. Important examples such as the rational, real, and

complex numbers are presented to illustrate how fields serve as foundational components in both pure and applied mathematics. The concept of field extensions is also introduced, paving the way for advanced topics.

Chapter 5: Homomorphisms and Isomorphisms

Building on the concepts of groups, rings, and fields, this chapter delves into the relationships between these structures through homomorphisms and isomorphisms. Homomorphisms are defined as structure-preserving maps that allow for the comparison of algebraic structures, while isomorphisms show when two structures are fundamentally the same. Theorems involving these concepts underscore their significance in classifying and understanding algebraic systems.

Chapter 6: Advanced Topics

The final chapters explore more complex and nuanced topics such as the correspondence between algebraic structures and their applications in other areas of mathematics, like linear algebra and number theory. It discusses topics such as Galois Theory, which connects field theory and group theory, and touches upon topics like modules and vector spaces, providing students with an expansive view of the relevance and applicability of abstract algebra.

Conclusion

Overall, "A First Course in Abstract Algebra" presents a logical and coherent journey through the fundamental principles of abstract algebra. It lays a robust foundation for further study, preparing students to engage with more complex mathematical theories and applications in their academic and professional pursuits. Through its clear exposition and systematic approach, the textbook not only imparts essential knowledge but also fosters a deeper appreciation for the beauty and utility of abstract algebra.

About the author

In the chapters focusing on John B. Fraleigh, the narrative unfolds against the backdrop of his illustrious career as a prominent mathematician. The text details his vital contributions to abstract algebra, a branch of mathematics that studies algebraic structures such as groups, rings, and fields. Fraleigh's academic journey and dedication to mathematics education are central themes, illustrating how he transformed complex algebraic concepts into accessible learning experiences.

Key to understanding Fraleigh's impact is his renowned textbook, "A First Course in Abstract Algebra," which is highlighted for its clarity and engaging style. It serves as an essential resource in undergraduate mathematics courses, enabling students to grasp intricate ideas through well-explained theories rather than rote memorization. This pedagogical approach not only aids in comprehension but also fosters a genuine appreciation for algebra, distinguishing Fraleigh's methods from traditional teaching practices that often prioritize procedural learning.

The chapters also explore his educational philosophy, which advocates for instilling a robust understanding of underlying concepts in students. This perspective aims to ignite a passion for mathematics, encouraging learners to engage with the material on a deeper level. Fraleigh's legacy within higher education is underlined by the many students and educators he has

influenced, cementing his role as a pivotal figure in the mathematics community.

In summary, the chapters depict John B. Fraleigh not only as a mathematician of notable repute but also as a dedicated educator whose work continues to resonate within the academic landscape of mathematics, shaping future generations' understanding and appreciation of the subject.

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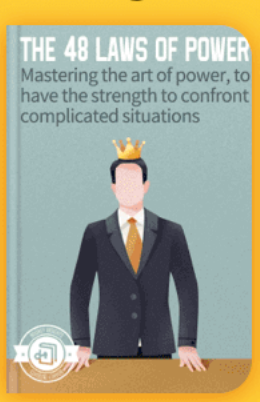

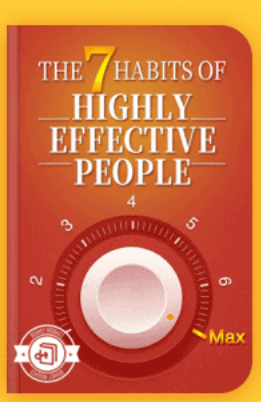

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Summary Content List

Chapter 1 Summary: 0 Sets and Relations

SECTION 0: SETS AND RELATIONS

On Definitions and the Notion of a Set

In mathematics, clear communication hinges on precise definitions. Certain foundational concepts, however, resist explicit definitions. Take "set," for instance; it introduces terms like "collection" that remain undefined. Mathematicians treat "set" as a primitive concept. A set S comprises distinct elements, marked by the notation $a \in S$ for an element a belonging to S. The empty set, denoted as $\emptyset$, is uniquely recognized. Sets can be detailed by either enumerating their elements or by characterizing their properties through set-builder notation, depicted as $\{x \mid P(x)\}$. It is essential to note that sets are well-defined; each object can be distinctly classified as either belonging to a set or not.

Boldface Convention

Within this section, terms and key concepts are presented in boldface to aid comprehension and retention. Familiarity with the ideas is prioritized over

exact memorization of definitions.

Basic Definitions and Notation

- **Subset:** A set B is considered a subset of another set A (notated as $B \subseteq A$) if every element within B is also found in A.
- **Improper and Proper Subset:** A set is an improper subset of itself, while all other subsets qualify as proper subsets. For example, if $S = \{1, 2, 3\}$, its subsets include the empty set as well as single-element and multi-element subsets.
- **Cartesian Product:** The Cartesian product $A \times B$ pairs each element from set A with each element from set B, resulting in ordered pairs defined as $\{(a, b) | a \in A, b \in B\}$.

Familiar Sets Notation

Different kinds of number sets are denoted as follows:
- $\mathbb{Z}$: The set of integers, which includes all whole numbers.
- $\mathbb{Q}$: The set of rational numbers, which are numbers expressible as fractions.
- $\mathbb{R}$: The set of real numbers, encompassing all rational and irrational numbers.
- $\mathbb{C}$: The set of complex numbers, integrating both real and imaginary components.

Relations Between Sets

A **relation** R between two sets A and B is fundamentally a subset of the Cartesian product $A \times B$. An example is the equality relation that determines when two elements x and y are identical. A **function** ϕ, pertaining to sets X and Y, is a specific type of relation where each $x \in X$ corresponds to one, and only one, ordered pair in ϕ.

Cardinality

The **cardinality** of a set X is indicated by $|X|$. Two sets are said to possess the same cardinality if a bijective (one-to-one and onto) correspondence exists between them.

Definition of One-to-One and Onto Functions

- A function is termed **injective** if it ensures that distinct x values map to unique y values, preventing any overlap.
- A function is considered **surjective** if every element in the codomain is mapped by at least one element from the domain.

Partitions and Equivalence Relations

- A **partition** refers to a collection of nonempty subsets wherein every element belongs to exactly one subset.
- An **equivalence relation** is characterized by three properties: reflexivity, symmetry, and transitivity, effectively building a bridge between elements.

Theorem on Equivalence Relations and Partitions

It is significant to note that every equivalence relation on a set naturally leads to a corresponding partition of that set, and conversely, each partition gives rise to a unique equivalence relation.

Exercises

To conclude this section, various exercises are presented to reinforce the key ideas surrounding sets, relations, functions, and cardinality. Tasks encompass establishing equivalence relations and analyzing different proposed sets and functions, ensuring a thorough understanding of foundational concepts in set theory.

Chapter 2 Summary: I Groups and Subgroups

Chapter 2 Summary: Groups and Subgroups

Section 1: Binary Operations

The chapter begins with a transition from basic arithmetic to the more abstract realm of algebra, where the focus shifts to binary operations. These operations take pairs of elements from a set S and map them into S itself. Denoted by an asterisk $*$, when applied to elements a and b in S, we express this as $a * b$. Familiar examples include traditional addition and multiplication, but the discussion also extends to more specialized operations, such as those applied to matrices. An important concept introduced is closure; an operation is said to be closed if performing it on elements from a subset results in an outcome that remains within that subset.

Section 2: Groups

The concept of a group is introduced as a pivotal framework in algebra. A group is defined as a set G equipped with a binary operation $*$ that

adheres to three key properties: associativity, the presence of an identity element (an element that does not change others when used in the operation), and the existence of an inverse for each element (an element that, when combined with another, yields the identity). The chapter provides examples of groups, such as the set of real numbers under addition \((R, +) \) and the nonzero real numbers under multiplication \((R*, \cdot) \). Groups can be categorized into abelian (where the order of operations does not matter) and non-abelian (where it does).

Section 3: Abelian Examples

Within abelian groups, the chapter highlights \(Z_n \), the group of integers under addition modulo \(n \). It also explains cyclic groups, which are generated by a single element called a generator, and emphasizes the significance of element order and subgroups linked to divisors of the group's order. A classic example is the set of all integers under addition, where the structure of subgroups is dictated by common divisors.

Section 4: Nonabelian Examples

In contrast, nonabelian groups are explored through the lens of permutations, particularly the symmetric group \(S_n \). These groups are characterized by their non-commutative nature—meaning the outcome of the operation depends on the order of the elements involved. The chapter delves into the

implications of having non-commuting elements and examines the intricate structure of permutations that constitute these groups.

Section 5: Subgroups

The chapter defines subgroups as subsets \(H \) of a group \(G \) that meet specific criteria. It discusses essential conditions for \(H \) to qualify as a subgroup, illuminating examples such as finite cyclic groups and providing illustrations that include dihedral groups, which represent symmetries of a polygon.

Section 6: Cyclic Groups

Focusing more narrowly on cyclic groups, the text asserts that every subgroup of a cyclic group is also cyclic. It elaborates on the order of cyclic groups and their generators, clarifying how integers \(r \) and \(s \) related through their greatest common divisors contribute to the formation of subgroups.

Section 7: Generating Sets and Cayley Digraphs

To visualize the abstract structures of groups, the chapter introduces Cayley digraphs. These directed graphs represent group elements as vertices, with directed edges indicating possible operations (or products) between them.

The properties of these digraphs are explored, showcasing their utility in revealing group characteristics, subgroup relationships, and structural insights.

This chapter provides a comprehensive foundation for the study of binary operations and group theory. Through the examination of both cyclic and non-cyclic groups, enriched with various examples and theoretical discussions, the reader develops a solid understanding of group structures and their significance in algebra.

Chapter 3 Summary: II Structure of Groups

PART II: Structure of Groups

Section 8: Groups of Permutations

This section delves into the concept of homomorphisms, which are functions mapping elements between groups while preserving their operation structure. Specifically, a function $\phi: G \to G'$ is considered a homomorphism if it satisfies the condition $\phi(ab) = \phi(a)\phi(b)$ for all elements a and b within the group G. This concept is pivotal in understanding the relationship between different algebraic structures, much like how certain mathematical functions can represent various forms, such as between real numbers and their unit circle equivalents.

Two key characteristics accompany homomorphisms: the **kernel**, which comprises all elements in G that map to the identity in G', and the **image**, which consists of all outputs from the mapping function.

Properties of homomorphisms include the preservation of identity elements, maintaining the structure of inverse elements, and retaining subgroups' properties. A significant result in group theory, **Cayley's Theorem**, asserts that every group can be represented as a subgroup of permutations, illustrating the profound connection between abstract algebra and combinatorial structures.

Section 9: Finitely Generated Abelian Groups

This section introduces **finitely generated abelian groups**, which can be formed through **direct products** of simpler groups, akin to forming new geometrical figures from basic shapes. The **Structure Theorems** elucidate that any finitely generated abelian group can be decomposed into a direct product of cyclic groups, represented in two forms: primary factorization and invariant factorization. For example, groups with an order of 360 can be expressed through various combinations of these cyclic group constructs, showcasing the breadth of possibilities when analyzing group structures.

Section 10: Cosets and the Theorem of Lagrange

In this section, the concept of **cosets** emerges, providing a way to organize elements within a group into distinct partitions through left and right cosets. This organization illustrates how subgroups can be viewed within larger groups. **Lagrange's Theorem** is a cornerstone result stating that the order (size) of any subgroup H within a finite group G must divide the order of G. This theorem leads to fascinating implications regarding the existence and nature of subgroups. The **index of subgroups**, denoted $(G:H)$, quantifies how many distinct cosets exist for H in G, establishing a mathematical relationship with the groups' orders and further deepening our understanding of group structure.

Section 11: Plane Isometries

The focus shifts to transformations in the plane, where certain actions, known as **isometries**, preserve distances and angles in \(\mathbb{R}^2 \). These transformations include translations, rotations, reflections, and glide reflections, each maintaining the geometric integrity of figures. A pivotal result in this area is the theorem stating that finite groups of plane isometries are isomorphic to three specific groups: the **Klein 4-group**, cyclic groups \(Z_n \), or dihedral groups \(D_n \). This classification reveals the symmetry underlying various isometric transformations. Additionally, the section explores **infinite groups** arising from repeating patterns, such as **frieze groups** and **wallpaper groups**, which categorize symmetries based on their unique types.

Exercises

The chapter concludes with a series of exercises designed to reinforce understanding of the definitions, theorems, and examples discussed in the sections. These activities encourage further exploration of the material, inviting readers to apply their knowledge to solve problems and deepen their understanding of group structures and properties.

Chapter 4: III Homomorphisms and Factor Groups

Summary of Chapter 4: Factor Groups and Group Actions

Section 12: Factor Groups

This section introduces factor groups, obtained by partitioning a group \(G \) into cosets of a normal subgroup \(H \). A subgroup is considered normal if its left cosets and right cosets are equivalent, which is crucial for the multiplication of these sets to be well-defined. The chapter illustrates these concepts using examples, particularly highlighting the subgroup of even permutations in symmetric groups. It also presents a significant theorem: the multiplication of left cosets is well-defined only if \(H \) is a normal subgroup, which is central to understanding the structure of factor groups.

Further, the chapter defines the factor group \(G/H \), which comprises the cosets of \(H \) in \(G \) and inherits the group structure from \(G \). Key properties demonstrate that \(G/H \) maintains group characteristics; notably, if \(G \) is abelian, then \(H \) is also normal.

The discussion transitions to the **Fundamental Homomorphism Theorem**, linking homomorphisms to factor groups. This theorem states that every group homomorphism leads to a factor group that preserves essential

structural properties of the original group. The concepts of normal subgroups and inner automorphisms are also elaborated, emphasizing that various definitions of normality are equivalent, with the center of any non-abelian group serving as an important normal subgroup.

Exercises included in this section encourage readers to actively engage with factor group computations and the classification of groups, reinforcing theoretical understanding through practice.

Section 13: Factor-Group Computations and Simple Groups

In this section, the focus shifts to practical examples of factor group computations, particularly concerning simple groups—those groups that do not have non-trivial normal subgroups. The alternating groups serve as prime examples of simple groups, with the chapter explaining how any finite group can be decomposed into simple components, providing significant insight into the structure of groups.

Orbits and Group Actions introduce the concept of group actions—how groups operate on sets—and emphasize the importance of counting orbits, which is foundational for understanding the structure of finite groups.

The section culminates with **Burnside's Theorem**, a vital tool for counting the number of distinct arrangements, or orbits, under group actions. By applying this theorem to real-life symmetry problems, readers can better grasp its practical significance.

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Chapter 5 Summary: IV Advanced Group Theory

Chapter 5 Summary: A First Course in Abstract Algebra

SECTION 16: Isomorphism Theorems

The chapter opens with the First Isomorphism Theorem, establishing a critical connection in group theory: it asserts that the quotient group formed by a homomorphism correlates uniquely to its image. Understanding this theorem relies on grasping the role of normal subgroups, which are foundational in the progression of subsequent isomorphism theorems. Following this, the Second Isomorphism Theorem elaborates on the interactions between a subgroup and a normal subgroup, illustrating that their product creates a new group structure. The Third Isomorphism Theorem further examines the implications of normal subgroups, leading to the formation of additional quotient groups, thereby enriching our understanding of group interactions.

SECTION 17: Sylow Theorems

Next, the chapter introduces the Sylow Theorems, which extend Cauchy's earlier findings about finite groups to subgroups of prime power order. The first of these theorems guarantees that such subgroups exist, while the

second articulates that any two Sylow p-subgroups can be transformed into one another by conjugation. The third theorem quantifies these subgroups, revealing that their count must be congruent to one modulo p. Illustrative examples clarify these concepts, particularly within specific group orders, highlighting how the Sylow Theorems inform group classifications.

SECTION 18: Series of Groups

Building on these concepts, the chapter explores normal and subnormal series, which reveal the intricate structure between a group and its subgroups through sequenced series. The Jordan-Hölder theorem proves the uniqueness of composition series, which aids in assessing group solvability. Additionally, the discussion of ascending central series offers further insights into the layered structure of groups, providing a framework for understanding how groups can be systematically analyzed.

SECTION 19: Free Abelian Groups

The focus then shifts to free abelian groups, which are defined by their unique property that allows for linear combinations from a generating set. Theorem 19.5 asserts that any nonzero free abelian group can be represented as the direct sum of a specified number of integers, showcasing their structural character. Unique rank characterization highlights crucial properties of these groups and illustrates their interplay with subgroup

structures, providing clarity on their formation.

SECTION 20: Free Groups

The exploration continues with free groups, characterized by their generation from a set under specific conditions regarding word relations. Theorem 20.12 affirms that homomorphisms defined on free groups are uniquely determined by their generators. The examination of free groups paves the way to understanding their foundational role in constructing various other group types, including those formed through commutator subgroups.

SECTION 21: Group Presentations

Finally, the concept of group presentations emerges, defining groups by their generators and the relations that hold among them. This section provides diverse examples of how finite groups can be expressed through presentations, albeit with a hint of the complexities involved in identifying isomorphisms between different formulations. The discussions reflect the broad applicability of group presentations in algebra, opening pathways for future exploration in group classifications.

Overall, this chapter navigates through the intricate world of advanced group theory, encapsulating key theorems and concepts that not only facilitate deeper analysis of group structures but also stress their significance across

various mathematical contexts.

Chapter 6 Summary: V Rings and Fields

Summary of Chapter 6: Rings and Fields

Section 22: Rings and Fields

In this pivotal section, the concepts of *rings* and *fields* are introduced, essential structures in abstract algebra defined by their respective properties involving two binary operations: addition and multiplication. A *ring* $\langle R, +, \cdot \rangle$ comprises a set R where the addition operation forms an abelian group, multiplication is associative, and both operations adhere to the distributive laws. Common examples include the integers $\mathbb{Z}$, rational numbers $\mathbb{Q}$, real numbers $\mathbb{R}$, complex numbers $\mathbb{C}$, polynomial rings, and matrix rings. Historical contributions to ring theory from notable mathematicians, such as David Hilbert and Emmy Noether, highlight its significance and evolution.

Definitions and Basic Properties

Rings are defined by three key axioms: R_1 establishes addition as an

abelian group, \(R_2 \) asserts the associative property of multiplication, and \(R_3 \) embodies the distributive properties linking the two operations. Various subclasses of rings emerge from these definitions, including *commutative rings* (where multiplication is commutative), *rings with unity* (featuring a multiplicative identity), *division rings* (where every non-zero element has a multiplicative inverse), and *fields* (which are commutative division rings).

Homomorphisms and Isomorphisms

A *homomorphism* serves as a structure-preserving map between two rings, maintaining the operations' integrity. An *isomorphism* is a special type of homomorphism that is both bijective, confirming the structural equivalence of the two rings involved.

Multiplicative Questions: Fields

Fields emerge as a crucial concept, characterized as commutative division rings in which every non-zero element possesses a multiplicative inverse. Key examples include the fields of rational numbers \(\mathbb{Q} \), real numbers \(\mathbb{R} \), and finite fields like \(\mathbb{Z}_p \).

Integral Domains

An *integral domain* is defined as a commutative ring with unity devoid of zero divisors, linking smoothly to the concept of fields. This section examines the distinction between integral domains and fields through various examples, particularly within finite rings, clarifying the specific properties that elucidate this relationship.

Fermat’s and Euler’s Theorems

The chapter delves into significant results such as *Fermat’s Little Theorem*, which posits that if p is a prime number not dividing a, then $a^{p-1} \equiv 1 \mod p$. Euler expands this by stating that if a is relatively prime to n, then $a^{\phi(n)} \equiv 1 \mod n$, where ϕ denotes the Euler phi function, essential in number theory and applications in cryptography.

Application and Encryption

Concluding the chapter, the practical application of these abstract concepts is illustrated through the RSA encryption scheme, which employs large prime numbers to secure communications. This encryption technology leverages the computational complexity of factoring products of primes, demonstrating the profound relevance of rings and fields in modern cryptography.

Exercises

Finally, the chapter includes a variety of exercises designed to reinforce understanding of ring computations, properties of ideals, and the application of theorems within the broader context of ring and field theory. These exercises provide a practical framework for engaging with the foundational topics discussed in the chapter.

This summary presents a cohesive overview of the chapter, ensuring that essential terminology and concepts are maintained while elucidating their interconnections within the overarching structure of abstract algebra.

Chapter 7 Summary: VI Constructing Rings and Fields

Part VI: Constructing Rings and Fields

In this section, we explore the foundational concepts of rings and fields in abstract algebra, focusing on integral domains, polynomials, error-correcting codes, homomorphisms, and various types of ideals.

Section 26: The Field of Quotients of an Integral Domain

An integral domain D can be extended to create a field F, known as the field of quotients. This field consists of all fractions $\frac{a}{b}$ where a and b are elements of D and b is non-zero. A key example is the transformation of integers $\mathbb{Z}$ into the rational numbers $\mathbb{Q}$.

Construction Steps involve:

1. Defining the elements of the new field F.
2. Establishing the operations of addition and multiplication.
3. Confirming that F adheres to the axioms of a field.
4. Showing that D acts as a subring within F.

An equivalence relation is defined for pairs (a, b) in $D \times D$ to

ensure structural integrity, leading to verification of the basic arithmetic operations within the field.

Section 27: Rings of Polynomials

A polynomial over a ring R consists of a formal sum of terms $a_n x^n + a_{n-1} x^{n-1} + \ldots + a_0$, where coefficients are from R and are only non-zero for finitely many indices. This set of polynomials forms a ring denoted $R[x]$.

For example, the polynomial $x^2 + 1$ in $\mathbb{Z}_2[x]$ serves to illustrate how operations within this ring function, including how we add and multiply polynomials.

Section 28: Factorization of Polynomials over a Field

Understanding how to factor polynomials is crucial for identifying their zeros. The Division Algorithm allows for dividing polynomials, leading to the influential Factor Theorem, which posits that if $f(a) = 0$, then a is a root of the polynomial.

Key points in this context include analyzing the products of polynomials for their zeros under various ideal conditions, such as those imposed by prime and maximal ideals.

Section 29: Algebraic Coding Theory

Coding theory is essential for ensuring the reliable transmission of messages in the presence of errors. Here, codes in \(Z^k_2 \) are defined, where the minimum distance between code words determines the efficacy of error detection and correction. Linear codes are identified as subgroups, with metrics like Hamming distance guiding the potential for detecting errors.

As an example, a simple parity-check code provides a basic level of error detection, while more sophisticated methods involve pattern encoding via polynomials, highlighting the relationship between algebra and communication technology.

Section 30: Homomorphisms and Factor Rings
Homomorphisms establish connections between ring structures, where the kernel of a homomorphism is recognized as an ideal. This relationship is fundamental for creating factor rings. Notable theorems explore how homomorphisms relate pairs of rings through their ideals and mappings.

A vital concept introduced is that if \(N \) is an ideal in a ring \(R \), then the quotient \(R/N \) has a well-defined ring structure, effectively reducing the complexity of the original ring.

Section 31: Prime and Maximal Ideals
Maximal ideals are of particular interest as they are linked to the construction of fields. When using maximal ideals in the form \(R/M \)

(where \(M \) is the maximal ideal), this quotient results in a field. In contrast, prime ideals are crucial for ensuring that the quotient \(R/N \) becomes an integral domain, inhibiting any divisors of zero.

This distinction is significant for factorization processes and the overarching structure within both commutative and noncommutative rings, revealing real-world applications in algebraic theories.

Section 32: Noncommutative Examples

In noncommutative rings, multiplication does not adhere to the commutative law. A prime example is the matrix ring \(M_n(F) \), where \(F \) is a field. The structure of the quaternions expands our understanding of algebraic systems, challenging traditional field theory and prompting deeper exploration into the implications of noncommutativity.

Overall, these sections present a comprehensive framework for understanding the construction and interrelation of abstract algebraic structures, reinforcing the principles through rigorous definitions, examples, and applications that lay the groundwork for advanced algebraic concepts.

Chapter 8: VII Commutative Algebra

Summary of Chapter 8: A First Course in Abstract Algebra by John B. Fraleigh

PART VII: Commutative Algebra

This chapter delves into essential concepts within commutative algebra, focusing on its foundational elements and their implications in various mathematical fields.

Section 33: Vector Spaces

The exploration of vector spaces begins by defining them as collections of vectors over any field that adhere to specific axioms related to addition and scalar multiplication. Key properties such as linear independence—the concept that some vectors cannot be expressed as a combination of others—and the dimension, which signifies the number of vectors in a basis, are introduced. Notable examples include R^n (n-dimensional real space) and polynomial spaces like F[x].

Historically, the idea of vector spaces evolved significantly in the 19th century, influenced by mathematicians such as Hamilton, who worked on quaternions; Grassmann, who focused on multidimensional spaces; and Peano, who formalized mathematical logic. Key theorems underscore that finite-dimensional vector spaces possess bases of uniform size, affirming that all bases are equivalent in dimension and providing methods for their determination.

Section 34: Unique Factorization Domains (UFDs)

UFDs are defined as integral domains in which every non-zero element can be uniquely expressed as a product of irreducible (or prime) factors, up to reorder and multiplication by units. An important subclass of UFDs is the Principal Ideal Domain (PID), exemplified by the integers (Z) and polynomial rings (F[x]), establishing a framework where unique factorization is guaranteed.

Key theorems demonstrate the correlation between these structures, illustrating that every PID is a UFD, while also referencing significant historical mathematical discoveries, including the remarkable implications of unique factorization in Fermat's Last Theorem.

Section 35: Euclidean Domains

The chapter further defines Euclidean domains, which allow the execution of division with remainders, thereby enabling the calculation of greatest common divisors (gcds). A crucial aspect of Euclidean domains is that they fall under the umbrella of PIDs, affirming their classification as UFDs.

The segment wraps up with practical applications related to gcd computations, underscoring methods that can be automated, thus highlighting the computational aspect of abstract algebra that facilitates research and analysis.

Section 36: Number Theory

Transitioning to number theory, the chapter introduces Gaussian integers, a set of complex numbers where both the real and imaginary parts are integers. Each Gaussian integer has a norm, which helps determine its properties,

including which elements act as units or irreducibles.

A significant theorem, Fermat's Theorem on Sums of Squares, states that an odd prime can be expressed as the sum of two squares if and only if it is congruent to 1 modulo 4, further connecting abstract algebra with classical number theory.

Section 37: Algebraic Geometry

The study of algebraic geometry is introduced through the analysis of the common zeros of polynomial equations, leading to the concept of algebraic varieties and polynomial ring ideals. Key theorems like the Hilbert Basis Theorem assert that ideals in polynomial rings over Noetherian rings possess finite generating sets, thereby linking algebraic structure with geometric representation.

Discussions of how to manipulate polynomials and ideal bases pave the way for insights into geometric properties, reinforcing the connections between abstract algebra and geometry.

Section 38: Gröbner Bases for Ideals

Gröbner bases are defined as a critical tool in the realm of polynomial ideals,

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Chapter 9 Summary: VIII Extension Fields

Part VIII: Extension Fields

Section 39: Introduction to Extension Fields

The core objective of this section is to establish that every nonconstant polynomial must have a zero, a fundamental aspect of algebra. A field E is referred to as an extension field of another field F when F is a subset of E.

Kronecker's Theorem is a pivotal theorem stating that for any field F and a nonconstant polynomial $f(x)$ in $F[x]$, there exists an extension field E of F and an element α within E such that $f(\alpha) = 0$. This theorem highlights an important mathematical insight introduced by Leopold Kronecker, who focused on the possibility of constructing mathematical entities, particularly by adding roots of irreducible polynomials to fields.

To construct extension fields, the proof involves the evaluation homomorphism ϕ_α. Specifically, it showcases that E, represented as $F[x]/\langle p(x) \rangle$, contains a zero of the polynomial $p(x)$.

Several examples illustrate the concept: for instance, in Example 39.4, the real numbers $\mathbb{R}$ serve as an extension of the rationals $\mathbb{Q}$, demonstrating how $f(x) = x^2 + 1$ has zeros in the complex numbers. Example 39.5 further supports this with the polynomial $f(x) = x^4 - 5x^2 + 6$ defined over $\mathbb{Q}$.

In the realm of field theory, elements within E can be classified as either algebraic or transcendental. An algebraic element α meets the requirement of satisfying a non-zero polynomial in $F[x]$, while a transcendental element does not. The irreducible polynomial of α is the unique monic polynomial having α as a root.

When α is algebraic over F, the field $F(\alpha)$ becomes a simple extension of F. There is a theorem stating that every element β in $F(\alpha)$ can be uniquely expressed with respect to a basis composed of $\{1, \alpha, \alpha^2, \ldots, \alpha^{n-1}\}$, where n is the degree of α over F. This representation implies that $F(\alpha)$ functions as a vector space of dimension n.

Section 40: Algebraic Extensions

This section introduces the concept of **algebraic extensions**. An extension field E over a base field F qualifies as algebraic if every element in

E is algebraic relative to F. A **finite extension** is defined by its finite dimension n over F, and its degree is denoted as $[E : F] = n$. A significant theorem reflects that any finite extension E of F is inherently algebraic, with its degree ascertainable through the size of its basis.

Diving into the structure of finite fields, we note that the non-zero elements of such fields establish a cyclic group under multiplication. A compelling result states that for each prime p and positive integer n, there exists a unique finite field, denoted as $GF(p^n)$, containing p^n elements.

Furthermore, a theorem elucidates the polynomial $x^{pn} - x$ possesses pn distinct zeros in an algebraically closed field, such as Z_p.

Section 41: Geometric Constructions

The exploration of **constructible numbers** reveals that a real number α is termed constructible if it can be derived through classic constructions involving a straightedge and compass, starting from a segment of unit length. Constructible numbers adhere to specific rules; if α and β are constructible, then $\alpha \pm \beta$ and $\alpha \beta$ are also constructible.

However, there are notable impossibilities as outlined by theorems:

duplicating a cube, squaring a circle, and trisecting an angle cannot be achieved using these classical geometric methods. This realization, initially not proven by Greek mathematicians, was made clear through advancements in field theory.

Section 42: Finite Fields

This section delves deeper into the **structure of finite fields**, confirming that the multiplicative group of their non-zero elements is cyclic. Each finite field of order \(p^n \) is unique. A remarkable theorem posits that for every finite field of characteristic \(p \), the number of elements is precisely \(p^n \) for a given integer \(n \).

Finite fields hold substantial applications across various mathematical domains, including coding theory, combinatorial designs, and algebraic topology, showcasing their importance in both theoretical and practical contexts.

The section concludes with a series of exercises designed to reinforce understanding and allow for verification of the concepts and theorems introduced throughout the sections.

Chapter 10 Summary: IX Galois Theory

Chapter 10 Summary: Insolvability of the Quintic

Introduction to the Problem

This chapter explores a pivotal question in algebra concerning the solvability of polynomial equations. It focuses on quintic polynomials, or fifth-degree equations, demonstrating that they cannot universally be solved using radicals—a method that works for quadratic, cubic, and quartic polynomials. This crucial finding was pioneered by the mathematician Niels Henrik Abel in the early 19th century, reshaping the understanding of polynomial solutions.

Extensions by Radicals

To understand the concept of solvability, it's important to define what is meant by an extension by radicals. An extension K of a field F qualifies as such if the elements within K can be derived from F through a series of finite operations entailing addition, multiplication, and the extraction of roots (i.e., radicals). A polynomial $f(x)$ is deemed solvable by radicals if its splitting field—the field that contains all roots of the polynomial—can be constructed through these operations.

Understanding the Galois Group

The chapter asserts a vital connection between polynomials and group theory: a polynomial is solvable by radicals if and only if its Galois group is a solvable group. The Galois group, named after the mathematician Évariste Galois, describes the symmetries of the polynomial's roots. If the Galois group is not solvable, it indicates that the polynomial itself cannot be simplified into solutions expressed in radicals.

Construction of a Non-Solvable Polynomial

To illustrate this concept, the chapter presents a specific quintic polynomial, \(f(x) = 2x^5 - 5x^4 + 5 \). This polynomial is shown to be irreducible over the field of rational numbers \(\mathbb{Q} \) and has a Galois group that is isomorphic to the symmetric group \(S_5 \), which comprises all possible permutations of five elements. It is noteworthy that this polynomial exhibits three real roots and two complex roots, exemplifying the intricate structure of its roots and their relationships to the Galois group.

The Galois Group's Properties

A crucial theorem elaborated in the chapter states that any Galois group that contains both a transposition (a simple swap of two elements) and a 5-cycle

(a permutation of five elements in a single loop) must be the complete symmetric group S_5. This means that the associated Galois group $G(K/\mathbb{Q})$ is non-solvable, reinforcing the conclusion that the quintic polynomial cannot be solved through radicals.

Abel’s Theorems

The chapter delves into the historical context surrounding Abel's groundbreaking proof, shedding light on the contributions of notable mathematicians who came before him. Their collective work led to the realization that not all fifth-degree polynomials could be resolved using radical expressions, marking a fundamental shift in the field of algebra.

Conclusion

In summation, this chapter emphasizes the vast array of quintic polynomials that are inherently unsolvable by radicals, establishing a clear connection between polynomial characteristics, their roots, and corresponding Galois groups. Abel's contributions are underscored as a major milestone in the evolution of algebra, laying the groundwork for modern mathematical theory and explaining the limitations of classical methods in solving higher-degree equations.